

Functional limit theorems for the maxima of perturbed random walks and divergent perpetuities in the M_1 -topology

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Abstract

Let $(\xi_1, \eta_1), (\xi_2, \eta_2), \dots$ be a sequence of i.i.d. two-dimensional random vectors. In the earlier article Iksanov and Pilipenko (2014) weak convergence in the J_1 -topology on the Skorokhod space of $n^{-1/2} \max_{0 \leq k \leq n} (\xi_1 + \dots + \xi_k + \eta_{k+1})$ was proved under the assumption that contributions of $\max_{0 \leq k \leq n} (\xi_1 + \dots + \xi_k)$ and $\max_{1 \leq k \leq n} \eta_k$ to the limit are comparable and that $n^{-1/2}(\xi_1 + \dots + \xi_{[n.]})$ is attracted to a Brownian motion. In the present paper, we continue this line of research and investigate a more complicated situation when $\xi_1 + \dots + \xi_{[n.]}$, properly normalized without centering, is attracted to a centered stable Lévy process, a process with jumps. As a consequence, weak convergence normally holds in the M_1 -topology. We also provide sufficient conditions for the J_1 -convergence. For completeness, less interesting situations are discussed when one of the sequences $\max_{0 \leq k \leq n} (\xi_1 + \dots + \xi_k)$ and $\max_{1 \leq k \leq n} \eta_k$ dominates the other. An application of our main results to divergent perpetuities with positive entries is given.

Key words: functional limit theorem; J_1 -topology; M_1 -topology; perpetuity; perturbed random walk

1 Introduction and results

Let $(\xi_k, \eta_k)_{k \in \mathbb{N}}$ be a sequence of i.i.d. two-dimensional random vectors with generic copy (ξ, η) . No condition is imposed on the dependence structure between ξ and η . Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Further, let $(S_n)_{n \in \mathbb{N}_0}$ be the zero-delayed ordinary random walk with increments ξ_n for $n \in \mathbb{N}$, i.e., $S_0 = 0$ and $S_n = \xi_1 + \dots + \xi_n$, $n \in \mathbb{N}$. Then define its perturbed variant $(T_n)_{n \in \mathbb{N}}$, that we call *perturbed random walk*, by

$$T_n := S_{n-1} + \eta_n, \quad n \in \mathbb{N}. \quad (1)$$

Recently it has become a rather popular object of research, see the recent book [9] for a survey and [1, 2, 8, 11, 16, 17, 18, 19, 20, 23, 25]. It is worth noting that sometimes in the literature the term ‘perturbed random walk’ was used to denote random sequences other than those defined in (1). See, for instance, [5, 6, 12, 14, 15, 21, 27] and Section 6 in [7].

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Denote by $D := D[0, \infty)$ the Skorokhod space of real-valued right-continuous functions which are defined on $[0, \infty)$ and have finite limits from the left at each positive point. Throughout the paper we assume that D is equipped with either the J_1 -topology or the M_1 -topology. We refer to [3, 13] and [26] for comprehensive accounts of the J_1 - and the M_1 -topologies, respectively. We write $\mathcal{M}_p(A)$ for the set of Radon point measures on a ‘nice’ space A . The $\mathcal{M}_p(A)$ is endowed with vague convergence. More information on these can be found in [22]. Throughout the paper $\xRightarrow{J_1}$ and $\xRightarrow{M_1}$ will mean weak convergence on the Skorokhod space D when endowed with the J_1 -topology and the M_1 -topology, respectively. The notation \Rightarrow without superscript is normally followed by a specification of the topology and the space involved. Finally, we write \xrightarrow{v} and \xrightarrow{P} to denote vague convergence and convergence in probability, respectively.

In the present paper we are interested in weak convergence on D of $\max_{0 \leq k \leq [n]} (S_k + \eta_{k+1})$, properly normalized without centering, as $n \rightarrow \infty$. It should not come as a surprise that the maxima exhibit three types of different behaviors depending on the asymptotic interplay of $A_n := \max_{0 \leq k \leq n} S_k$ and $B_n := \max_{1 \leq k \leq n+1} \eta_k$, namely, on whether (I) A_n dominates B_n ; (II) A_n is dominated by B_n ; (III) A_n and B_n are comparable.

Relying essentially upon findings in [11] three functional limit theorems for the maxima of perturbed random walks, properly rescaled without centering, were proved in [9] under the assumption that $\mathbb{E}\xi^2 < \infty$. Throughout the remainder of the paragraph we assume that the most interesting alternative (III) prevails. The situation treated in [11] was relatively simple because the limit process for $S_{[n]}/n^{1/2}$ was a Brownian motion, a process with continuous paths. As a consequence, the convergence took place in the J_1 -topology on D , and, more surprisingly, the contributions of (S_k) and (η_j) turned out asymptotically independent, despite the possible strong dependence of ξ and η . In the present paper we treat a more delicate case where the distribution of ξ belongs to the domain of attraction of an α -stable distribution, $\alpha \in (0, 2)$, so that the limit process for $S_{[n]}$, properly normalized, is an α -stable Lévy process. We shall show that the presence of jumps in the latter process destroys dramatically an idyllic picture pertaining to the Brownian motion scenario: the convergence typically holds in the weaker M_1 -topology on D , and the aforementioned asymptotic independence only occurs in some exceptional cases where ξ and η are themselves asymptotically independent in an appropriate sense.

Throughout the paper we assume that, as $x \rightarrow \infty$,

$$\mathbb{P}\{|\xi| > x\} \sim x^{-\alpha}\ell(x) \quad (2)$$

and that

$$\mathbb{P}\{\xi > x\} \sim c_1 \mathbb{P}\{|\xi| > x\}, \quad \mathbb{P}\{-\xi > x\} \sim c_2 \mathbb{P}\{|\xi| > x\} \quad (3)$$

for some $\alpha \in (0, 2)$, some ℓ slowly varying at ∞ , some nonnegative c_1 and c_2 summing up to one. The assumptions mean that the distribution of ξ belongs to the domain of attraction of an α -stable distribution. To ensure weak convergence of $S_{[n]}$ without centering we assume that $\mathbb{E}\xi = 0$ when $\alpha \in (1, 2)$ and that the distribution of ξ is symmetric when $\alpha = 1$. Then the classical Skorokhod theorem (Theorem 2.7 in [24]) tells us that

$$\frac{S_{[n]}}{a(n)} \xRightarrow{J_1} \mathcal{S}_\alpha(\cdot), \quad n \rightarrow \infty, \quad (4)$$

where $a(x)$ is a positive function satisfying $\lim_{x \rightarrow \infty} x \mathbb{P}\{|\xi| > a(x)\} = 1$ and $\mathcal{S}_\alpha := (\mathcal{S}_\alpha(t))_{t \geq 0}$ is an α -stable Lévy process with the characteristic function

$$\mathbb{E} \exp(iz \mathcal{S}_\alpha(1)) = \exp(|z|^\alpha (\Gamma(2 - \alpha)/(\alpha - 1)) (\cos(\pi\alpha/2) - i(c_1 - c_2) \sin(\pi\alpha/2) \text{sign} z)), \quad z \in \mathbb{R}$$

when $\alpha \in (0, 2)$, $\alpha \neq 1$, here, $\Gamma(\cdot)$ denotes the Euler gamma function, and

$$\mathbb{E} \exp(iz\mathcal{S}_\alpha(1)) = \exp(-2^{-1}\pi|z|), \quad z \in \mathbb{R}$$

when $\alpha = 1$.

Put $E := [-\infty, +\infty] \times [0, \infty] \setminus \{(0, 0)\}$. For a Radon measure ρ on E , let $N^{(\rho)} := \sum_k \varepsilon_{(\theta_k, i_k, j_k)}$ be a Poisson random measure on $[0, \infty) \times E$ with mean measure $\mathbb{L}\mathbb{E}\mathbb{B} \times \rho$, where $\varepsilon_{(t, x, y)}$ is the probability measure concentrated at $(t, x, y) \subset [0, \infty) \times E$, $\mathbb{L}\mathbb{E}\mathbb{B}$ is the Lebesgue measure on $[0, \infty)$.

Theorem 1.1 which is the main result of the present paper treats the most complicated situation (III) when the contributions of $\max_{0 \leq k \leq n} S_k$ and $\max_{1 \leq k \leq n+1} \eta_k$ to the asymptotic behavior of $\max_{0 \leq k \leq n} (S_k + \eta_{k+1})$ are comparable. At this point it is worth stressing that ξ and η are assumed arbitrarily dependent which makes the analysis nontrivial. We stipulate hereafter that the supremum over the empty set equals zero.

Theorem 1.1. *Suppose that conditions (2) and (3) hold, that $\mathbb{P}\{\eta > x\} \sim c\mathbb{P}\{|\xi| > x\}$ as $x \rightarrow \infty$, for some $c > 0$, and that*

$$x\mathbb{P}\left\{\frac{(\xi, \eta^+)}{a(x)} \in \cdot\right\} \xrightarrow{\nu} \nu, \quad x \rightarrow \infty \quad (5)$$

on $\mathcal{M}_p(E)$. Then

$$\frac{\max_{0 \leq k \leq [n \cdot]} (S_k + \eta_{k+1})}{a(n)} \xrightarrow{M_1} \sup_{0 \leq s \leq \cdot} \mathcal{S}_\alpha^*(s) \vee \sup_{\theta_k \leq \cdot} (\mathcal{S}_\alpha^*(\theta_k -) + j_k), \quad n \rightarrow \infty, \quad (6)$$

where (θ_k, i_k, j_k) are the atoms of a Poisson random measure $N^{(\nu)}$ and \mathcal{S}_α^* is a copy of \mathcal{S}_α whose Lévy-Itô representation is built upon the Poisson random measure $\sum_k \varepsilon_{(\theta_k, i_k)}$.

Under the additional assumption

$$\nu\{(x, y) : 0 < y < x\} = 0, \quad (7)$$

the convergence in (6) holds in the J_1 -topology on D .

We proceed with a number of remarks.

Remark 1.2. It is perhaps worth stating explicitly that $N^{(\nu)}(\cdot \times [-\infty, +\infty] \times \cdot) = \sum_k \varepsilon_{(\theta_k, j_k)}$ is a Poisson random measure on $[0, \infty) \times (0, \infty]$ with mean measure $\mathbb{L}\mathbb{E}\mathbb{B} \times \mu_c$, where μ_c is a measure on $(0, \infty]$ defined by

$$\mu_c((x, \infty]) = cx^{-\alpha}, \quad x > 0.$$

Analogously, $N^{(\nu)}(\cdot \times \cdot \times [0, \infty]) = \sum_k \varepsilon_{(\theta_k, i_k)}$ is a Poisson random measure on $[0, \infty) \times ([-\infty, +\infty] \setminus \{0\})$ with mean measure $\mathbb{L}\mathbb{E}\mathbb{B} \times \nu^*$, where ν^* is the Lévy measure of \mathcal{S}_α given by

$$\nu^*((x, \infty]) = c_1 x^{-\alpha}, \quad \nu^*((-\infty, -x]) = c_2 x^{-\alpha}, \quad x > 0.$$

Remark 1.3. Condition (7) obviously holds if the measure ν is concentrated on the axes. This is the case whenever ξ and η are independent and also in many cases when these are dependent. For instance, take $\xi = |\log W|$ and $\eta = |\log(1 - W)|$ satisfying (5) for a random variable W taking values in $(0, 1)$ (details can be found in the proof of Theorem 1.1 in [10]).

Suppose now that $\eta = r\xi$ for some $r > 0$. Then the restriction of ν to the first quadrant concentrates on the line $y = rx$. Hence, condition (7) holds if, and only if, $r \geq 1$.

Let ρ be a positive random variable such that $\mathbb{P}\{\rho > x\} \sim x^{-\alpha}$ as $x \rightarrow \infty$ and ζ a random variable which is independent of ρ and takes values in $[-\pi, \pi)$. Setting $\xi = \rho \cos \zeta$ and $\eta = \rho \sin \zeta$ we obtain

$$\mathbb{P}\{\xi > x\} \sim (\mathbb{E}(\cos \zeta)^\alpha \mathbb{1}_{\{|\zeta| < \pi/2\}})x^{-\alpha}, \quad \mathbb{P}\{-\xi > x\} \sim (\mathbb{E}|\cos \zeta|^\alpha \mathbb{1}_{\{|\zeta| > \pi/2\}})x^{-\alpha}$$

as $x \rightarrow \infty$ by the Lebesgue dominated convergence theorem. Furthermore,

$$x\mathbb{P}\left\{\frac{(\xi, \eta^+)}{a(x)} \in \cdot\right\} \xrightarrow{v} \nu, \quad x \rightarrow \infty,$$

where $a(x) = (\mathbb{E}|\cos \zeta|^\alpha)^{1/\alpha} x^{1/\alpha}$ and ν is the image of the measure

$$\frac{\alpha \mathbb{E}(\cos \zeta)^\alpha \mathbb{1}_{\{\zeta \in (-\pi/2, \pi/2)\}}}{\mathbb{E}|\cos \zeta|^\alpha} \mathbb{1}_{(r, \varphi) \in (0, \infty) \times [0, \pi)} r^{-\alpha-1} dr \mathbb{P}\{\zeta \in d\varphi\}$$

under the mapping $(r, \varphi) \rightarrow (r \cos \varphi, r \sin \varphi)$. Condition (7) is equivalent to $\mathbb{P}\{\zeta \in (0, \pi/4)\} = 0$.

Remark 1.4. Weak convergence of nondecreasing processes in the M_1 -topology is not as strong as it might appear. Actually, it is equivalent to weak convergence of finite-dimensional distributions just because a sequence of nondecreasing processes is always tight on D equipped with the M_1 -topology. This follows from the fact that the M_1 -oscillation

$$\omega_\delta(f) := \sup_{t_1 \leq t \leq t_2, 0 \leq t_2 - t_1 \leq \delta} M(f(t_1), f(t), f(t_2))$$

of a nondecreasing function f equals zero, where $M(x_1, x_2, x_3) := 0$, if $x_2 \in [x_1, x_3]$, and $:= \min(|x_2 - x_1|, |x_3 - x_2|)$, otherwise.

Remark 1.5. From a look at Theorem 2.1 underlying the proof of Theorem 1.1 it should be clear that a counterpart of Theorem 1.1 also holds when replacing the input vectors $(\xi_k, \eta_k)_{k \in \mathbb{N}}$ with arrays $(\xi_k^{(n)}, \eta_k^{(n)})_{k \in \mathbb{N}}$ for each $n \in \mathbb{N}$. We however refrain from formulating such a generalization, for we are not aware of any potential applications of such a result.

Propositions 1.6 and 1.8 given next are concerned with the simpler situations (I) and (II), respectively.

Proposition 1.6. *Suppose that conditions (2) and (3) hold and that*

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}\{\eta > x\}}{\mathbb{P}\{|\xi| > x\}} = 0. \quad (8)$$

Then

$$\frac{\max_{0 \leq k \leq [n]} (S_k + \eta_{k+1})}{a(n)} \xrightarrow{J_1} \sup_{0 \leq s \leq \cdot} \mathcal{S}_\alpha(s), \quad n \rightarrow \infty. \quad (9)$$

Remark 1.7. When $\alpha \in (0, 1)$ and $c_1 = 0$, the right-hand side in (9) is the zero function because \mathcal{S}_α is then the negative of an α -stable subordinator (recall that a subordinator is a nondecreasing Lévy process). In this setting there are two possibilities: either $\sup_{k \geq 0} (S_k + \eta_{k+1}) < \infty$ a.s. or $\sup_{k \geq 0} (S_k + \eta_{k+1}) = \infty$ a.s. Plainly, if the first alternative prevails, much more than (9) can be said, namely, $\max_{0 \leq k \leq [n]} (S_k + \eta_{k+1})/r_n$ converges to the zero function in the J_1 -topology on D for any positive sequence (r_n) diverging to ∞ .

Now we intend to give examples showing that either of possibilities can hold. By Theorem 2.1 in [1], the supremum of $S_k + \eta_{k+1}$ is finite a.s. if, and only if,

$$\int_{(0,\infty)} \frac{x}{\int_0^x \mathbb{P}\{-\xi > y\} dy} d\mathbb{P}\{\xi \leq y\} < \infty \quad \text{and} \quad \int_{(0,\infty)} \frac{x}{\int_0^x \mathbb{P}\{-\xi > y\} dy} d\mathbb{P}\{\eta \leq y\} < \infty \quad (10)$$

If $\mathbb{P}\{-\xi > x\} \sim x^{-\alpha}$, $\mathbb{P}\{\xi > x\} \sim x^{-\alpha}(\log x)^{-2}$ as $x \rightarrow \infty$ and $\mathbb{E}(\eta^+)^{\alpha} < \infty$, then both inequalities in (10) hold, whereas if $\mathbb{P}\{-\xi > x\} \sim x^{-\alpha}$, $\mathbb{P}\{\xi > x\} \sim x^{-\alpha}(\log x)^{-1}$ as $x \rightarrow \infty$, then the first integral in (10) diverges.

Proposition 1.8. *Suppose that conditions (2) and (3) hold, that*

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}\{\eta > x\}}{\mathbb{P}\{|\xi| > x\}} = \infty \quad (11)$$

and that $\mathbb{P}\{\eta > x\}$ is regularly varying at ∞ of index $-\beta$ (necessarily $\beta \in (0, \alpha]$). Let $b(x)$ be a positive function which satisfies $\lim_{x \rightarrow \infty} x\mathbb{P}\{\eta > b(x)\} = 1$. Then

$$\frac{\max_{0 \leq k \leq [n]} (S_k + \eta_{k+1})}{b(n)} \xrightarrow{J_1} \sup_{\theta_k \leq \cdot} j_k, \quad n \rightarrow \infty, \quad (12)$$

where (θ_k, j_k) are the atoms of a Poisson random measure on $[0, \infty) \times (0, \infty]$ with mean measure $\mathbb{L}\mathbb{E}\mathbb{B} \times \mu$, where μ is a measure on $(0, \infty]$ defined by

$$\mu((x, \infty]) = x^{-\beta}, \quad x > 0.$$

Whenever the random series $\sum_{k \geq 0} e^{T_{k+1}}$ converges a.s., its sum is called *perpetuity* due to its occurrence in the realm of insurance and finance as a sum of discounted payment streams. When the random series diverges, it is natural to investigate weak convergence on D of its partial sums, properly rescaled, as the number of summands becomes large. Some results of this flavor can be found in [4] and [9] (in these works many references to earlier one-dimensional results can be found). Here, we prove functional limit theorems in the situations that remained untouched.

Theorem 1.9. *In the settings of Theorem 1.1 and Propositions 1.6 and 1.8 functional limit theorems hold with $\log \left(\sum_{k=0}^{[n]} e^{T_{k+1}} \right)$ replacing $\max_{0 \leq k \leq [n]} T_{k+1}$.*

2 Proof of Theorem 1.1

For each $n \in \mathbb{N}$, let $(x_i^{(n)}, y_i^{(n)})_{i \in \mathbb{N}}$ be a sequence of \mathbb{R}^2 -valued vectors. Put $S_0^{(n)} := 0$,

$$S_k^{(n)} := \sum_{i=1}^k x_i^{(n)}, \quad k \in \mathbb{N}, \quad T_k^{(n)} := S_k^{(n)} + y_{k+1}^{(n)}, \quad k \in \mathbb{N}_0$$

and then define the piecewise constant functions

$$f_n(t) := \sum_{k \geq 0} S_k^{(n)} \mathbb{1}_{[\frac{k}{n}, \frac{k+1}{n})}(t), \quad g_n(t) := \max_{0 \leq k \leq [nt]} T_k^{(n)}, \quad t \geq 0$$

where $\mathbb{1}_A(x) = 1$ if $x \in A$ and $= 0$, otherwise.

To proceed, we have to recall the notation $E = [-\infty, +\infty] \times [0, \infty] \setminus \{(0, 0)\}$. The proof of Theorem 1.1 is essentially based on the following deterministic result along with the continuous mapping theorem.

Theorem 2.1. Let $f_0 \in D$ and $\nu_0 = \sum_k \varepsilon_{(t_k, x_k, y_k)}$ be a Radon measure on $[0, \infty) \times E$ satisfying $\nu_0(\{0\} \times E) = 0$ and $t_k \neq t_j$ for $k \neq j$. Suppose that

$$\lim_{n \rightarrow \infty} f_n = f_0 \quad (13)$$

in the J_1 -topology on D and that

$$\nu_n := \sum_{i \geq 1} \varepsilon_{(i/n, x_i^{(n)}, y_i^{(n)})} \mathbb{1}_{\{y_i^{(n)} > 0\}} \xrightarrow{v} \nu_0, \quad n \rightarrow \infty \quad (14)$$

on $\mathcal{M}_p([0, \infty) \times E)$. Then

$$\lim_{n \rightarrow \infty} g_n = g_0 := \sup_{0 \leq s \leq \cdot} f_0(s) \vee \sup_{t_k \leq \cdot} (f_0(t_k -) + y_k)$$

in the M_1 -topology on D . This convergence holds in the J_1 -topology on D under the additional assumption

$$\nu_0([0, \infty) \times \{(x, y) : 0 < y < x\}) = 0. \quad (15)$$

Remark 2.2. Suppose that f_0 is continuous and that the set of points (t_k, y_k) with $y_k > 0$ is dense in $[0, \infty)$. Then

$$g_0(\cdot) = \sup_{t_k \leq \cdot} (f_0(t_k) + y_k).$$

Furthermore, condition (15) holds automatically, and condition (14) is equivalent to

$$\sum_{i \geq 1} \varepsilon_{(i/n, y_i^{(n)})} \mathbb{1}_{\{y_i^{(n)} > 0\}} \xrightarrow{v} \sum_k \varepsilon_{(t_k, y_k)} \quad n \rightarrow \infty.$$

on $\mathcal{M}_p([0, \infty) \times (0, \infty])$. This is the setting of Theorem 1.3 in [11].

Remark 2.3. Here, we discuss the necessity of condition (15) for the J_1 -convergence. Suppose that in the setting of Theorem 2.1 there exists $k \in \mathbb{N}$ such that $0 < y_k < x_k$, so that condition (15) does not hold. Now we give an example in which the J_1 -convergence in Theorem 2.1 fails to hold. With $x_i^{(n)} = y_i^{(n)} = 0$ for $i \neq [n/2]$, $x_{[n/2]}^{(n)} = 2$ and $y_{[n/2]}^{(n)} = 1$ we have $f_n(t) = 2 \mathbb{1}_{[[n/2]/n, \infty)}(t)$ and

$$g_n(t) = \begin{cases} 0, & t < \frac{[n/2]-1}{n}, \\ 1, & \frac{[n/2]-1}{n} \leq t < \frac{[n/2]}{n}, \\ 2, & \frac{[n/2]}{n} \leq t. \end{cases}$$

Plainly, condition (14) holds with $\nu_0 = \varepsilon_{(1/2, 2, 1)}$. Setting $f_0(t) = g(t) := 2 \mathbb{1}_{[1/2, \infty)}(t)$ we conclude that $\lim_{n \rightarrow \infty} f_n = f_0$ in the J_1 -topology and $\lim_{n \rightarrow \infty} g_n = g$ in the M_1 -topology. On the other hand, g_n has a jump of magnitude 1 at point $[n/2]/n$. Furthermore, this magnitude does not converge to 2, the size of the limit jump at point $1/2$. This precludes the J_1 -convergence.

Lemma 2.4 given next collects known criteria for the convergence of nondecreasing functions in the J_1 - and M_1 -topologies. For part (a), see Corollary 12.5.1 and Lemma 12.5.1 in [26]. While one implication of part (b) is standard, the other follows from Theorem 2.15 on p. 342 and Lemma 2.22 on p. 343 in [13].

Lemma 2.4. Let $(h_n)_{n \in \mathbb{N}_0}$ be a sequence of nondecreasing functions in D .

(a) $\lim_{n \rightarrow \infty} h_n = h_0$ in the M_1 -topology on D if, and only if, $h_n(t)$ converges to $h_0(t)$ for each t in a dense subset of continuity points of h_0 including zero.

(b) $\lim_{n \rightarrow \infty} h_n = h_0$ in the J_1 -topology on D if, and only if, $\lim_{n \rightarrow \infty} h_n = h_0$ in the M_1 -topology on D and for any discontinuity point s of h_0 there exists a sequence $(s_n)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} s_n = s$,

$$\lim_{n \rightarrow \infty} h_n(s_n-) = h(s-) \quad \text{and} \quad \lim_{n \rightarrow \infty} h_n(s_n) = h(s).$$

Proof of Theorem 2.1. We start by showing that $g_0 \in D$. Since g_0 is nondecreasing, it has finite limits from the left on $(0, \infty)$. Using right-continuity of f_0 we obtain

$$\begin{aligned} g_0(t) &\leq \lim_{\delta \rightarrow 0+} g_0(t + \delta) = \lim_{\delta \rightarrow 0+} \left(\sup_{0 \leq s < t + \delta} f_0(s) \vee \sup_{t_k \leq t + \delta} (f_0(t_k-) + y_k) \right) \\ &= \sup_{0 \leq s \leq t} f_0(s) \vee \lim_{\delta \rightarrow 0+} \sup_{t_k \leq t + \delta} (f_0(t_k-) + y_k) \\ &\leq \sup_{0 \leq s \leq t} f_0(s) \vee \sup_{t_k \leq t} (f_0(t_k-) + y_k) \vee \lim_{\delta \rightarrow 0+} \sup_{t < t_k \leq t + \delta} (f_0(t_k-) + y_k) \\ &\leq \sup_{0 \leq s \leq t} f_0(s) \vee \sup_{t_k \leq t} (f_0(t_k-) + y_k) \vee \left(\lim_{\delta \rightarrow 0+} \sup_{t < t_k \leq t + \delta} f_0(t_k-) + \lim_{\delta \rightarrow 0+} \sup_{t < t_k \leq t + \delta} y_k \right) \\ &= \sup_{0 \leq s \leq t} f_0(s) \vee \sup_{t_k \leq t} (f_0(t_k-) + y_k) \vee (f_0(t) + 0) = g_0(t) \end{aligned}$$

for any $t > 0$ which proves right-continuity of g_0 .

PROOF OF THE M_1 -CONVERGENCE. Since $f_0, g_0 \in D$, they have at most countably many discontinuities. Hence, the set

$$K := \{T \geq 0 : \nu_0(\{T\} \times E) = 0; \quad T \text{ is a continuity point of } f_0 \text{ and continuity point of } g_0\}$$

is dense in $[0, \infty)$. Since g_n is nondecreasing for each $n \in \mathbb{N}$, according to Lemma 2.4 (a), it suffices to prove that

$$\lim_{n \rightarrow \infty} g_n(T) = g_0(T) \tag{16}$$

for all $T \in K$. Observe that $g_0(0) = 0$ as a consequence of $f_n(0) = f_0(0) = 0$ and $\nu_0(\{0\} \times E) = 0$. The last condition ensures that $g_n(0) = y_1^{(n)}$ converges to zero as $n \rightarrow \infty$. This proves that relation (16) holds for $T = 0$. Thus, in what follows we assume that $T \in K$ and $T > 0$.

Fix any such a T . There exists a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ that vanishes as $k \rightarrow \infty$ and such that its generic element denoted by ε is a continuity point of the nonincreasing function

$$x \mapsto \nu_0([0, T] \times [-\infty, +\infty] \times (x, \infty]),$$

so that $\nu_0([0, T] \times [-\infty, +\infty] \times \{\varepsilon\}) = 0$. Put $E_\varepsilon := [-\infty, +\infty] \times (\varepsilon, \infty]$. Condition (14) implies that $\nu_0([0, T] \times E_\varepsilon) = \nu_n([0, T] \times E_\varepsilon) = m$ for large enough n and some $m \in \mathbb{N}_0$, where the finiteness of m is secured by the fact that ν_0 is a Radon measure. The case $m = 0$ is trivial. Hence, in what follows we assume that $m \in \mathbb{N}$. Denote by $(\bar{t}_i, \bar{x}_i, \bar{y}_i)_{1 \leq i \leq m}$ an enumeration of the points of ν_0 in $[0, T] \times E_\varepsilon$ with

$$\bar{t}_1 < \bar{t}_2 < \dots < \bar{t}_m \tag{17}$$

and by $(\bar{t}_i^{(n)}, \bar{x}_i^{(n)}, \bar{y}_i^{(n)})_{1 \leq i \leq m}$ the analogous enumeration of the points ν_n in $[0, T] \times E_\varepsilon$. Note that $\bar{t}_1 > 0$ in view of the assumption $\nu_0(\{0\} \times E) = 0$, whereas the assumption $t_k \neq t_j$ for $k \neq j$ ensures that the inequalities in (17) are strict. Then

$$\lim_{n \rightarrow \infty} \sum_{i=1}^m (|\bar{t}_i^{(n)} - \bar{t}_i| + |\bar{x}_i^{(n)} - \bar{x}_i| + |\bar{y}_i^{(n)} - \bar{y}_i|) = 0. \tag{18}$$

Later on we shall need the following relation

$$f_n(\bar{t}_i^{(n)} - 1/n) = f_n(\bar{t}_i^{(n)} -) \rightarrow f_0(\bar{t}_i -), \quad n \rightarrow \infty \quad (19)$$

for $i = 1, \dots, m$. To prove it, fix any $i = 1, \dots, m$ and assume that \bar{t}_i is a discontinuity point of f_0 . Then condition (13) in combination with $f_n(\bar{t}_i^{(n)}) - f_n(\bar{t}_i^{(n)} -) = \bar{x}_i^{(n)} \rightarrow \bar{x}_i \neq 0$ as $n \rightarrow \infty$ entails $\bar{x}_i = f_0(\bar{t}_i) - f_0(\bar{t}_i -)$ and (19) (see the proof of Proposition 2.1 on p. 337 in [13]). If \bar{t}_i is a continuity point of f_0 , (19) holds trivially. Arguing similarly, we also obtain

$$\lim_{n \rightarrow \infty} \max_{t \in [0, \bar{t}_i^{(n)} - 2/n]} f_n(t) = \sup_{t \in [0, \bar{t}_i]} f_0(t) \quad (20)$$

for $i = 1, \dots, m$.

We first work with functions $g_{n,\varepsilon}$ and $g_{0,\varepsilon}$ which are counterparts of g_n and g_0 based on the restrictions of ν_n and ν_0 to $[0, T] \times E_\varepsilon$. Put

$$A_{n,T} := \{j \in \mathbb{N}_0 : 0 \leq j \leq [nT], (j+1)/n \neq \bar{t}_i^{(n)} \text{ for } i = 1, \dots, m\}.$$

Now we are ready to write a basic decomposition

$$\begin{aligned} g_{n,\varepsilon}(T) &:= \max_{0 \leq i \leq [nT]} (x_1^{(n)} + \dots + x_i^{(n)} + y_{i+1}^{(n)} \mathbb{1}_{\{y_{i+1}^{(n)} > \varepsilon\}}) \\ &= \max_{i \in A_{n,T}} f_n(i/n) \vee \max_{1 \leq k \leq m} (f_n(\bar{t}_k^{(n)} - 1/n) + \bar{y}_k^{(n)}) \\ &= \max_{0 \leq i \leq [nT]} f_n(i/n) \vee \max_{1 \leq k \leq m} (f_n(\bar{t}_k^{(n)} - 1/n) + \bar{y}_k^{(n)}) \\ &= \max_{t \in [0, T]} f_n(t) \vee \max_{1 \leq k \leq m} (f_n(\bar{t}_k^{(n)} - 1/n) + \bar{y}_k^{(n)}), \end{aligned} \quad (21)$$

the third equality following from the fact that, for integer $i \in [0, [nT]]$ such that $i/n = \bar{t}_k^{(n)} - 1/n$ for some $k = 1, \dots, m$, we have $f_n(i/n) < \max_{1 \leq k \leq m} (f_n(\bar{t}_k^{(n)} - 1/n) + \bar{y}_k^{(n)})$ because all the $\bar{y}_k^{(n)}$ are positive.

It is convenient to state the following known result as a lemma, for it will be used twice in the subsequent proof.

Lemma 2.5. *Let s_0 be a continuity point of f_0 and $(s_n)_{n \in \mathbb{N}}$ a sequence of positive numbers converging to s_0 as $n \rightarrow \infty$. Then $\lim_{n \rightarrow \infty} \sup_{t \in [0, s_n]} f_n(t) = \max_{t \in [0, s_0]} f_0(t)$.*

Proof. We first observe that $\sup_{t \in [0, s_0]} f_0(t) = \max_{t \in [0, s_0]} f_0(t)$ because s_0 is a continuity point of f_0 (hence, of the supremum). It is well-known (and easily checked) that (13) entails

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq \cdot} f_n(t) = \sup_{0 \leq t \leq \cdot} f_0(t) \quad (22)$$

in the J_1 -topology on D . In particular, $\lim_{n \rightarrow \infty} \sup_{t \in [0, s_n]} f_n(t) = \max_{t \in [0, s_0]} f_0(t)$. \square

Recalling that T is a continuity point of f_0 and using Lemma 2.5 with $s_n = T$ for all $n \in \mathbb{N}_0$ we infer

$$\lim_{n \rightarrow \infty} \max_{t \in [0, T]} f_n(t) = \sup_{t \in [0, T]} f_0(t)$$

and thereupon

$$\lim_{n \rightarrow \infty} g_{n,\varepsilon}(T) = \sup_{s \in [0, T]} f_0(s) \vee \sup_{\bar{t}_k \leq T} (f_0(\bar{t}_k -) + \bar{y}_k) := g_{0,\varepsilon}(T) \quad (23)$$

having utilized (18) and (19) for the second supremum.

Further, we claim that

$$\sup_{t \geq 0} |g_0(t) - g_{0,\varepsilon}(t)| \leq \varepsilon \quad \text{and} \quad \sup_{t \geq 0} |g_n(t) - g_{n,\varepsilon}(t)| \leq \varepsilon. \quad (24)$$

We only prove the first inequality, the proof of the second being analogous and simpler. Write

$$\begin{aligned} |g_0(t) - g_{0,\varepsilon}(t)| &= g_0(t) - g_{0,\varepsilon}(t) = \sup_{s \in [0, t]} f_0(s) \vee \sup_{t_k \leq t} (f_0(t_k -) + y_k) \\ &\quad - \sup_{s \in [0, t]} f_0(s) \vee \sup_{\bar{t}_k \leq t} (f_0(\bar{t}_k -) + \bar{y}_k) \end{aligned}$$

for all $t \geq 0$. There are two possibilities: either

$$\sup_{t_k \leq t} (f_0(t_k -) + y_k) = \sup_{t_k \leq t, t_k \neq \bar{t}_k} (f_0(t_k -) + y_k) \vee \sup_{\bar{t}_k \leq t} (f_0(\bar{t}_k -) + \bar{y}_k) = \sup_{\bar{t}_k \leq t} (f_0(\bar{t}_k -) + \bar{y}_k)$$

in which case $|g_0(t) - g_{0,\varepsilon}(t)| = 0$ for all $t \geq 0$, i.e., the first inequality in (24) holds, or

$$\sup_{t_k \leq t} (f_0(t_k -) + y_k) = \sup_{t_k \leq t, t_k \neq \bar{t}_k} (f_0(t_k -) + y_k).$$

Observe that

$$\sup_{t_k \leq t, t_k \neq \bar{t}_k} (f_0(t_k -) + y_k) \leq \sup_{s \in [0, t]} f_0(s) + \varepsilon$$

as a consequence of $y_k \leq \varepsilon$ for all $k \in \mathbb{N}$ such that $t_k \neq \bar{t}_k$, and that

$$\sup_{s \in [0, t]} f_0(s) \vee \sup_{\bar{t}_k \leq t} (f_0(\bar{t}_k -) + \bar{y}_k) \geq \sup_{s \in [0, t]} f_0(s).$$

Hence, the first inequality in (24) holds in this case, too.

It remains to note that

$$\begin{aligned} |g_n(T) - g_0(T)| &\leq |g_n(T) - g_{n,\varepsilon}(T)| + |g_{n,\varepsilon}(T) - g_{0,\varepsilon}(T)| + |g_{0,\varepsilon}(T) - g_0(T)| \\ &\leq 2\varepsilon + |g_{n,\varepsilon}(T) - g_{0,\varepsilon}(T)| \end{aligned}$$

and then first let n tend to ∞ and use (23), and then let ε go to zero through the sequence (ε_k) .

This shows that $\lim_{n \rightarrow \infty} g_n(T) = g_0(T)$. The proof of the M_1 -convergence is complete.

PROOF OF THE J_1 -CONVERGENCE. We intend to prove that whenever \bar{s} is a discontinuity point of $g_{0,\varepsilon}$ there is a sequence $(s_n)_{n \in \mathbb{N}}$ converging to \bar{s} for which

$$\lim_{n \rightarrow \infty} g_{n,\varepsilon}(s_n) = g_{0,\varepsilon}(\bar{s}) \quad \text{and} \quad \lim_{n \rightarrow \infty} g_{n,\varepsilon}(s_n -) = g_{0,\varepsilon}(\bar{s} -). \quad (25)$$

Now we explain that (25) entails

$$\lim_{n \rightarrow \infty} g_n = g_0 \quad (26)$$

in the J_1 -topology on D , which is the desired result. From the first part of the proof we know that $\lim_{n \rightarrow \infty} g_{n,\varepsilon} = g_{0,\varepsilon}$ in the M_1 -topology on D . Thus, if (25) holds, we conclude that

$$\lim_{n \rightarrow \infty} g_{n,\varepsilon} = g_{0,\varepsilon} \quad (27)$$

in the J_1 -topology on D by Lemma 2.4(b). Let $r \in [0, T]$ be a continuity point of g_0 , where $T \in K$ (see the first part of the proof for the definition of K). In order to prove (26) it suffices to show that $\lim_{n \rightarrow \infty} g_n = g_0$ in the J_1 -topology on $D[0, r]$ or, equivalently, that $\lim_{n \rightarrow \infty} \rho(g_n, g_0) = 0$ where ρ is the standard Skorokhod metric on $[0, r]$. Since r is also a continuity point of $g_{0,\varepsilon}$, relation (27) ensures that $\lim_{n \rightarrow \infty} \rho(g_{n,\varepsilon}, g_{0,\varepsilon}) = 0$. We proceed by writing

$$\begin{aligned} \rho(g_n, g_0) &\leq \rho(g_n, g_{n,\varepsilon}) + \rho(g_{n,\varepsilon}, g_{0,\varepsilon}) + \rho(g_{0,\varepsilon}, g_0) \leq \sup_{0 \leq t \leq r} |g_n(t) - g_{n,\varepsilon}(t)| + \rho(g_{n,\varepsilon}, g_{0,\varepsilon}) \\ &+ \sup_{0 \leq t \leq r} |g_{0,\varepsilon}(t) - g_0(t)| \leq 2\varepsilon + \rho(g_{n,\varepsilon}, g_{0,\varepsilon}) \end{aligned}$$

having utilized the fact that ρ is dominated by the uniform metric on $[0, r]$ for the penultimate inequality and (24) for the last. Now sending $n \rightarrow \infty$ and then letting ε approach zero through the sequence (ε_k) proves $\lim_{n \rightarrow \infty} \rho(g_n, g_0) = 0$ and thereupon (26).

Passing to the proof of (25) we consider two cases.

CASE 1: \bar{s} is a discontinuity point of $g_{0,\varepsilon}$ and a continuity point of f_0 .

We claim that $\bar{s} = \bar{t}_k$ for some $k = 1, \dots, m$. Indeed, if $g_{0,\varepsilon}(\bar{s}) = \sup_{t \in [0, \bar{s}]} f_0(t)$, then \bar{s} is a continuity point of $g_{0,\varepsilon}$, a contradiction. Thus, we must have $g_{0,\varepsilon}(\bar{s}) = \max_{\bar{t}_j \leq \bar{s}} (f_0(\bar{t}_j-) + \bar{y}_j)$. The points $\bar{t}_1, \dots, \bar{t}_m$ are the only discontinuities of $x \mapsto \max_{\bar{t}_j \leq x} (f_0(\bar{t}_j-) + \bar{y}_j)$ on $[0, \infty)$. Therefore, $\bar{s} = \bar{t}_k$ for some $k = 1, \dots, m$, as claimed.

With this k , set $s_n = \bar{t}_k^{(n)} - 1/n$. Analogously to (21) we obtain

$$g_{n,\varepsilon}(\bar{t}_k^{(n)} - 1/n) = \max_{t \in [0, \bar{t}_k^{(n)} - 2/n]} f_n(t) \vee \max_{1 \leq j \leq k} (f_n(\bar{t}_j^{(n)} - 1/n) + \bar{y}_j^{(n)}) \quad (28)$$

and

$$g_{n,\varepsilon}((\bar{t}_k^{(n)} - 1/n) -) = g_{n,\varepsilon}(\bar{t}_k^{(n)} - 2/n) = \max_{t \in [0, \bar{t}_k^{(n)} - 2/n]} f_n(t) \vee \max_{1 \leq j \leq k-1} (f_n(\bar{t}_j^{(n)} - 1/n) + \bar{y}_j^{(n)}). \quad (29)$$

We shall now show

$$\lim_{n \rightarrow \infty} g_{n,\varepsilon}(\bar{t}_k^{(n)} - 1/n) = \sup_{t \in [0, \bar{t}_k]} f_0(t) \vee \max_{\bar{t}_j \leq \bar{t}_k} (f_0(\bar{t}_j-) + \bar{y}_j) = g_{0,\varepsilon}(\bar{t}_k)$$

and

$$\lim_{n \rightarrow \infty} g_{n,\varepsilon}((\bar{t}_k^{(n)} - 1/n) -) = \sup_{t \in [0, \bar{t}_k]} f_0(t) \vee \max_{\bar{t}_j < \bar{t}_k} (f_0(\bar{t}_j-) + \bar{y}_j) = g_{0,\varepsilon}(\bar{t}_k-).$$

Indeed, while the limit relations

$$\lim_{n \rightarrow \infty} \max_{1 \leq j \leq k} (f_n(\bar{t}_j^{(n)} - 1/n) + \bar{y}_j^{(n)}) = \max_{\bar{t}_j \leq \bar{t}_k} (f_0(\bar{t}_j-) + \bar{y}_j) \quad (30)$$

and

$$\lim_{n \rightarrow \infty} \max_{1 \leq j \leq k-1} (f_n(\bar{t}_j^{(n)} - 1/n) + \bar{y}_j^{(n)}) = \max_{\bar{t}_j < \bar{t}_k} (f_0(\bar{t}_j-) + \bar{y}_j) \quad (31)$$

are secured by (18) and (19), the limit relation

$$\lim_{n \rightarrow \infty} \max_{t \in [0, \bar{t}_k^{(n)} - 2/n]} f_n(t) = \sup_{t \in [0, \bar{t}_k]} f_0(t)$$

holds in view of Lemma 2.5 with $s_n = \bar{t}_k^{(n)} - 1/n$ for $n \in \mathbb{N}$ and $s_0 = \bar{t}_k$. Thus, formula (25) has been proved in Case 1.

CASE 2: \bar{s} is a discontinuity point of both $g_{0,\varepsilon}$ and f_0 .

SUBCASE 2.1: $\bar{s} = \bar{t}_k$ for some $k = 1, \dots, m$. We intend to check that (25) holds with $s_n = t_k^{(n)} - 1/n$. Using formulae (28) and (29) and recalling (20), (30) and (31) we infer

$$\lim_{n \rightarrow \infty} g_{n,\varepsilon}((\bar{t}_k^{(n)} - 1/n) -) = \sup_{t \in [0, \bar{t}_k)} f_0(t) \vee \max_{\bar{t}_j < \bar{t}_k} (f_0(\bar{t}_j -) + \bar{y}_j) = g_{0,\varepsilon}(\bar{t}_k -)$$

and

$$\lim_{n \rightarrow \infty} g_{n,\varepsilon}(\bar{t}_k^{(n)} - 1/n) = \sup_{t \in [0, \bar{t}_k)} f_0(t) \vee \max_{\bar{t}_j \leq \bar{t}_k} (f_0(\bar{t}_j -) + \bar{y}_j).$$

Since

$$f_0(\bar{t}_k) = f_0(\bar{t}_k -) + \bar{x}_k \leq f_0(\bar{t}_k -) + \bar{y}_k.$$

in view of (15), we conclude that

$$\sup_{t \in [0, \bar{t}_k)} f_0(t) \vee \max_{\bar{t}_j \leq \bar{t}_k} (f_0(\bar{t}_j -) + \bar{y}_j) = \sup_{t \in [0, \bar{t}_k]} f_0(t) \vee \max_{\bar{t}_j \leq \bar{t}_k} (f_0(\bar{t}_j -) + \bar{y}_j) = g_{0,\varepsilon}(\bar{t}_k),$$

thereby finishing the proof of (25) in this subcase.

SUBCASE 2.2: $\bar{s} \notin \{\bar{t}_1, \dots, \bar{t}_m\}$. Let r be a continuity point of f_0 satisfying $r > \bar{s}$. Recall that (13) entails (22). Hence, there is a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of continuous strictly increasing functions of $[0, r]$ onto $[0, r]$ such that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, r]} |\lambda_n(t) - t| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup_{s \in [0, r]} \left| \sup_{t \in [0, \lambda_n(s)]} f_0(t) - \sup_{t \in [0, s]} f_n(t) \right| = 0.$$

In particular, $\lim_{n \rightarrow \infty} \sup_{t \in [0, s_n]} f_n(t) = \sup_{t \in [0, \bar{s})} f_0(t)$ and $\lim_{n \rightarrow \infty} \sup_{t \in [0, s_n]} f_n(t) = \sup_{t \in [0, \bar{s}]} f_0(t)$, where $s_n := \lambda_n(\bar{s})$. We shall show that (25) holds with this choice of s_n . To this end, it only remains to note that

$$\lim_{n \rightarrow \infty} \max_{\bar{t}_j^{(n)} < s_n} (f_n(\bar{t}_j^{(n)}) + \bar{y}_j^{(n)}) = \lim_{n \rightarrow \infty} \max_{\bar{t}_j^{(n)} \leq s_n} (f_n(\bar{t}_j^{(n)}) + \bar{y}_j^{(n)}) = \max_{\bar{t}_j \leq \bar{s}} (f_0(\bar{t}_j -) + \bar{y}_j) = \max_{\bar{t}_j < \bar{s}} (f_0(\bar{t}_j -) + \bar{y}_j)$$

as a consequence of $\bar{s} \notin \{\bar{t}_1, \dots, \bar{t}_m\}$ and (18). Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} g_{n,\varepsilon}(s_n -) &= \lim_{n \rightarrow \infty} \sup_{t \in [0, s_n)} f_n(t) \vee \max_{\bar{t}_j^{(n)} < s_n} (f_n(\bar{t}_j^{(n)}) + \bar{y}_j^{(n)}) \\ &= \sup_{t \in [0, \bar{s})} f_0(t) \vee \max_{\bar{t}_j < \bar{s}} (f_0(\bar{t}_j) + \bar{y}_j^{(n)}) = g_{0,\varepsilon}(\bar{s} -) \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} g_{n,\varepsilon}(s_n) &= \lim_{n \rightarrow \infty} \sup_{t \in [0, s_n]} f_n(t) \vee \max_{\bar{t}_j^{(n)} \leq s_n} (f_n(\bar{t}_j^{(n)}) + \bar{y}_j^{(n)}) \\ &= \sup_{t \in [0, \bar{s}]} f_0(t) \vee \max_{\bar{t}_j \leq \bar{s}} (f_0(\bar{t}_j) + \bar{y}_j^{(n)}) = g_{0,\varepsilon}(\bar{s}) \end{aligned}$$

which proves (25).

The proof of Theorem 2.1 is complete. \square

Proof of Theorem 1.1. By Corollary 6.1 on p. 183 in [22], condition (5) entails

$$\sum_{l \geq 1} \varepsilon_{(l/n, \xi_l/a(n), \eta_l^+/a(n))} \Rightarrow \sum_k \varepsilon_{(\theta_k, i_k, j_k)}, \quad n \rightarrow \infty$$

on $\mathcal{M}_p([0, \infty) \times E)$ and thereupon

$$\left(\sum_{l \geq 1} \mathbb{1}_{\{\xi_l \neq 0\}} \varepsilon_{(l/n, \xi_l/a(n))}, \sum_{l \geq 1} \varepsilon_{(l/n, \xi_l/a(n), \eta_l^+/a(n))} \right) \Rightarrow \left(\sum_k \mathbb{1}_{\{i_k \neq 0\}} \varepsilon_{(\theta_k, i_k)}, \sum_k \varepsilon_{(\theta_k, i_k, j_k)} \right) \quad (32)$$

as $n \rightarrow \infty$ on $\mathcal{M}_p([0, \infty) \times ([-\infty, +\infty] \setminus \{0\})) \times \mathcal{M}_p([0, \infty) \times E)$ because the first coordinates are just the restrictions of the second from $[0, \infty) \times E$ on $[0, \infty) \times ([-\infty, +\infty] \setminus \{0\})$.

In the proof of Corollary 7.1 on p. 218 in [22] it is shown that the convergence of the first coordinates in (32) implies $S_{[n \cdot]}/a(n) \xrightarrow{J_1} \mathcal{S}_\alpha(\cdot)$ as $n \rightarrow \infty$. Starting with *full* relation (32) exactly the same reasoning leads to the conclusion

$$\left(\frac{S_{[n \cdot]}}{a(n)}, \sum_{l \geq 1} \varepsilon_{(l/n, \xi_l/a(n), \eta_l^+/a(n))} \right) \Rightarrow \left(\mathcal{S}_\alpha^*(\cdot), \sum_k \varepsilon_{(\theta_k, i_k, j_k)} \right), \quad n \rightarrow \infty$$

or, equivalently,

$$\left(\frac{S_{[n \cdot]}}{a(n)}, \sum_{l \geq 1} \varepsilon_{(l/n, \xi_l/a(n), \eta_l/a(n))} \mathbb{1}_{\{\eta_l > 0\}} \right) \Rightarrow \left(\mathcal{S}_\alpha^*(\cdot), \sum_k \varepsilon_{(\theta_k, i_k, j_k)} \right), \quad n \rightarrow \infty$$

in the product topology on $D \times \mathcal{M}_p([0, \infty) \times E)$. By the Skorokhod representation theorem there are versions which converge a.s. Retaining the original notation for these versions we want to apply Theorem 2.1 with $f_n(\cdot) = S_{[n \cdot]}/a(n)$, $f_0 = \mathcal{S}_\alpha^*$, $\nu_n = \sum_{l \geq 1} \varepsilon_{\{l/n, \xi_l/a(n), \eta_l/a(n)\}} \mathbb{1}_{\{\eta_l > 0\}}$ and $\nu_0 = N^{(\nu)} = \sum_k \varepsilon_{(\theta_k, i_k, j_k)}$. We already know that conditions (13) and (14) are fulfilled a.s. It is obvious that $N^{(\nu)}(\{0\} \times E) = 0$ a.s. In order to show that $N^{(\nu)}$ does not have clustered jumps a.s. i.e., $\theta_k \neq \theta_j$ for $k \neq j$ a.s., it suffices to check this property for $N^{(\nu)}([0, T] \times [-\infty, +\infty] \times (\delta, \infty]) \cap \cdot$ with $T > 0$ and $\varepsilon > 0$ fixed. This is done on p. 223 in [22]. Hence Theorem 2.1 is indeed applicable with our choice of f_n and ν_n , and (6) follows. \square

3 Proofs of Propositions 1.6 and 1.8 and Theorem 1.9

Proof of Proposition 1.6. Fix any $T > 0$. Note that (8) entails

$$\lim_{x \rightarrow \infty} x \mathbb{P}\{\eta > \varepsilon a(x)\} = 0 \quad (33)$$

for all $\varepsilon > 0$ because $a(x)$ is regularly varying at ∞ (of index $1/\alpha$). Since, for all $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}\left\{ \sup_{0 \leq s \leq T} \eta_{[ns]+1} > \varepsilon a(n) \right\} &= 1 - \left(\mathbb{P}\{\eta \leq \varepsilon a(n)\} \right)^{[nT]+1} \\ &\leq ([nT] + 1) \mathbb{P}\{\eta > \varepsilon a(n)\} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ in view of (33), we infer

$$\frac{\sup_{0 \leq s \leq T} \eta_{[ns]+1}}{a(n)} \xrightarrow{P} 0, \quad n \rightarrow \infty.$$

This in combination with (4) enables us to conclude that

$$\frac{S_{[n]} + \eta_{[n]+1}}{a(n)} \xRightarrow{J_1} \mathcal{S}_\alpha(\cdot), \quad n \rightarrow \infty$$

by Slutsky's lemma. Relation (9) now follows by the continuous mapping theorem because the supremum functional is continuous in the J_1 -topology. \square

Proof of Proposition 1.8. To begin with, we note that $\lim_{n \rightarrow \infty} (b(n)/a(n)) = \infty$ as a consequence of (11). Consequently,

$$S_{[n]}/b(n) \xRightarrow{J_1} \Xi(\cdot)$$

in view of (4), where $\Xi(t) := 0$ for $t \geq 0$. Further, according to Theorem 3.6 on p. 62 in combination with Corollary 6.1 on p. 183 in [22], regular variation of $\mathbb{P}\{\eta > x\}$ ensures that

$$\sum_{k \geq 0} \varepsilon_{(k/n, \eta_{k+1}/b(n))} \mathbb{1}_{\{\eta_{k+1} > 0\}} \Rightarrow N := \sum_k \varepsilon_{(\theta_k, j_k)}, \quad n \rightarrow \infty$$

on $\mathcal{M}_p([0, \infty) \times (0, \infty])$ and thereupon

$$\left(S_{[n]}/b(n), \sum_{k \geq 0} \mathbb{1}_{\{\eta_{k+1} > 0\}} \varepsilon_{(k/n, \eta_{k+1}/b(n))} \right) \Rightarrow (\Xi(\cdot), N), \quad n \rightarrow \infty$$

on $D \times \mathcal{M}_p([0, \infty) \times (0, \infty])$ equipped with the product topology. Arguing as in the proof of Theorem 1.1 we obtain (12) by an application of Remark 2.2 with $f_n(\cdot) = S_{[n]}/b(n)$, $f_0 = \Xi$, $\nu_n = \sum_{k \geq 0} \varepsilon_{(k/n, \eta_{k+1}/b(n))} \mathbb{1}_{\{\eta_{k+1} > 0\}}$ and $\nu_0 = N$. The condition $N((a, b) \times (0, \infty]) \geq 1$ a.s. whenever $0 < a < b$ required in Remark 2.2 holds because $\mu((0, \infty]) = \infty$. \square

Proof of Theorem 1.9. The limit relations of Theorem 1.1 and Propositions 1.6 and 1.8 can be written in a unified form as

$$\frac{\max_{0 \leq k \leq [n]} T_{k+1}}{c(n)} \Rightarrow X(\cdot), \quad n \rightarrow \infty$$

in the J_1 - or the M_1 -topology on D . Using this limit relation together with the inequality

$$\max_{0 \leq k \leq n} T_{k+1} \leq \log \left(\sum_{k=0}^n e^{T_{k+1}} \right) \leq \log(n+1) + \max_{0 \leq k \leq n} T_{k+1}$$

and the fact that $\lim_{n \rightarrow \infty} (\log n / c(n)) = 0$ we arrive at the desired conclusion

$$\frac{\log \sum_{k=0}^{[n]} e^{T_{k+1}}}{c(n)} \Rightarrow X(\cdot), \quad n \rightarrow \infty$$

in the J_1 - or M_1 -topology on D .

Since the limit process X is nonnegative a.s., the result remains true on replacing \log with \log^+ . \square

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